MATH 5061 Lecture 5 (Feb 10)

[Problem Set 3 is posted, due on Mar 3.]

Last time ...... differential form & d, volume form / orientation / integration

Recall: (M^n, g) Riemannian manifold

smooth n-manifold

pos. definite symm. (0,2)-tensor

An affine connection $\nabla : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$

a vector space / $\mathbb{R}$ and a $C^\infty(M)$-module

$$(X, Y) \xrightarrow{\nabla} \nabla_X Y$$

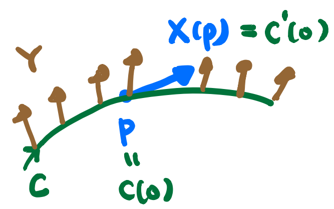
s.t. (i) ∇ is bilinear over $\mathbb{R}$ $\quad \forall X, Y \in \Gamma(TM)$

(ii) $\nabla_{fX} Y = f \nabla_X Y$ (tensorial in X) $\quad \forall f \in C^\infty(M)$

(iii) $\nabla_X (fY) = X(f) Y + f \nabla_X Y$ (Liebniz rule in Y)

Lemma: $\nabla_X Y(p)$ depends only on $X(p)$ and the values of Y along ANY curve tangent to X at p.

Proof: Locally in coordinates,



$$X = \sum_i a_i \frac{\partial}{\partial x^i} \quad ; \quad Y = \sum_j b_j \frac{\partial}{\partial x^j}$$

$$\nabla_X Y = \nabla_{\sum_i a_i \frac{\partial}{\partial x^i}} \Big(\sum_j b_j \frac{\partial}{\partial x^j} \Big) \overset{(i)}{\underset{(ii)}{=}} \sum_i a_i \nabla_{\frac{\partial}{\partial x^i}} \Big(\sum_j b_j \frac{\partial}{\partial x^j} \Big)$$

vector field

$$\overset{(i)}{=} \sum_{i,j} a_i \nabla_{\frac{\partial}{\partial x^i}} \Big(b_j \frac{\partial}{\partial x^j} \Big) \overset{(iii)}{=} \sum_{i,j} a_i \frac{\partial b_j}{\partial x^i} \frac{\partial}{\partial x^j} + \sum_{i,j} a_i b_j \boxed{\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}}$$

Write $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x^k}$, where Γ_{ij}^k = Christoffel symbols of ∇

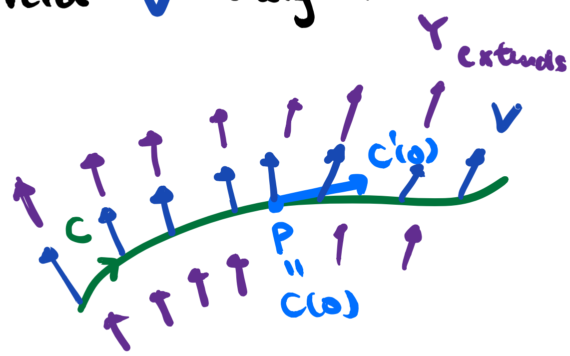
$$\nabla_X Y = \sum_k X(b_k) \frac{\partial}{\partial x^k} + \sum_{i,j,k} \Gamma_{ij}^k a_i b_j \frac{\partial}{\partial x^k}$$

$$= \sum_k \Big[\underbrace{X(b_k)}_{\substack{\text{depends only on} \\ X(p) \text{ and} \\ Y = b_k \text{ along a curve } c}} + \sum_{i,j} \underbrace{\Gamma_{ij}^k a_i b_j}_{\substack{\text{depends only} \\ \text{on } X(p), Y(p)}} \Big] \frac{\partial}{\partial x^k}$$

at p :

$\blacksquare$

This Lemma provides a way to define the notion of "Covariant derivative" of a vector field V along a curve $C(t)$ on M :



$$\underset{\text{at } p}{\frac{DV}{dt}} := \underset{\text{depends only on } c'(0) \text{ and } V.}{\nabla_{c'(t)} Y}$$

where Y is any smooth (local) extension of V

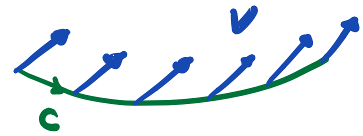
Prop: (1) $\frac{D}{dt}(V+W) = \frac{DV}{dt} + \frac{DW}{dt}$ $\quad \forall\, V, W \in \Gamma(TM)$.

(2) $\frac{D}{dt}(fV) = \frac{df}{dt} V + f \frac{DV}{dt}$ $\quad \forall\, f \in C^\infty(M)$

Proof: Follows from (i) – (iii) of ∇.

$\blacksquare$

Def^{n}: A vector field V along a curve C is **parallel** if $\boxed{\frac{DV}{dt} \equiv 0}$



Locally, write $V = \sum_i a_i \frac{\partial}{\partial x^i}$

$$C(t) = (C_1(t), \dots, C_n(t)) \quad ; \quad C'(t) = \sum_j C_j'(t) \frac{\partial}{\partial x^j}$$

So, V is parallel iff

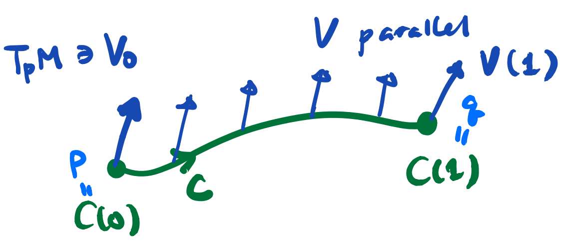
$$0 \equiv \frac{DV}{dt} = \nabla_{C'(t)} V = \nabla_{\sum_j C_j' \frac{\partial}{\partial x^j}} \left(\sum_i a_i \frac{\partial}{\partial x^i} \right)$$

$$\Rightarrow \quad 0 \equiv \sum_k \left(\frac{da_k}{dt} + \sum_{i,j} \Gamma_{ij}^k C_j' a_i \right) \frac{\partial}{\partial x^k}$$

i.e. $$\underbrace{\frac{da_k}{dt}(t) + \sum_{i,j=1}^{n} \Gamma_{ij}^k (c(t))\, C_j'(t)\, a_i(t) = 0}_{1^{st} \text{ order linear ODE system.}}, \quad \forall\, k = 1, \dots, n$$

$\Rightarrow$ Longtime existence & uniqueness given any initial data $a_i(0)$

In other words, if $C : [0,1] \to M$ is a smooth curve joining $p = C(0)$ to $q = C(1)$. Given any $V_0 \in T_pM$, $\exists!$ parallel v.f. $V(t)$ along C s.t. $V(0) = V_0$



The map $$P : T_pM \longrightarrow T_qM$$
$$V_0 \longmapsto V(1)$$

is called **parallel transport from p to q along C**

Ex: P is a linear isomorphism.

Fact: On the same manifold M, the space of affine connections defined on M forms an affine space, ie

$$\mathrm{Conn}(M) := \left\{ \begin{matrix} \text{affine} \\ \text{connections} \\ \text{on } M \end{matrix} \right\} = T_0 + \underbrace{W}_{\text{vector space}}$$

Motto: For a Riemannian manifold (M, g), there is a "canonical" connection called Riemannian / Levi-Civita connection.

Fundamental Thm. for Riemannian Geometry

Given a Riem. mfd (M^n, g), $\exists!$ connection ∇ s.t. $\forall X, Y, Z \in \mathcal{T}(TM)$

(1) "Metric compatible" ("$\nabla g \equiv 0$")

$$X\big(g(Y,Z)\big) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

(2) "Torsion-free"

$$\nabla_X Y - \nabla_Y X = \underbrace{[X,Y]}_{\text{smooth structure}}$$

Proof: Suppose such a connection ∇ exists.

(1) implies:

$$\begin{aligned}
& X\big(g(Y,Z)\big) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\
+\ & Y\big(g(Z,X)\big) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X) \\
-\ & Z\big(g(X,Y)\big) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)
\end{aligned}$$

$$\text{R.H.S.} \overset{(2)}{=} g([X,Z],Y) + g([Y,Z],X) + g([X,Y],Z) + 2\,g(\boxed{\nabla_Y X}, Z)$$

Rearranging gives **Kozsul formula**:

$$g(\nabla_Y X, Z) = \frac{1}{2}\left\{ \begin{array}{l} X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ - g([X,Z],Y) - g([Y,Z],X) - g([X,Y],Z) \end{array} \right\}$$

This shows ∇ exists and is unique! $\square$

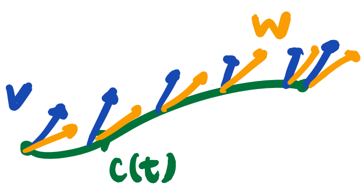
In local coord., $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x^k}$ where

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij})$$

Remarks:

- torsion-free $\Longleftrightarrow$ $\Gamma_{ij}^k = \Gamma_{ji}^k$

  Why? $\underbrace{\nabla_{\partial_i}\partial_j}_{\sum_k \Gamma_{ij}^k \partial_k} - \underbrace{\nabla_{\partial_j}\partial_i}_{\sum_k \Gamma_{ji}^k \partial_k} = [\partial_i, \partial_j] = 0$



- For covariant derivatives along a curve:

$$\frac{d}{dt}\big(g(V,W)\big) = g\left(\frac{DV}{dt}, W\right) + g\left(V, \frac{DW}{dt}\right)$$

In particular, if V, W are parallel, then $g(V,W) \equiv$ const. along c

So, V parallel $\Rightarrow$ $\|V\|^2 := g(V,V) \equiv$ const.

FROM NOW ON, WE ASSUME

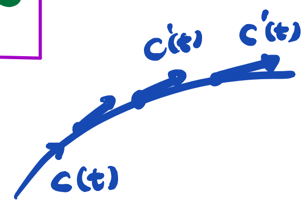
$\underbrace{(M^n, g)}_{\text{Riem mfd}}$ equipped with $\underbrace{\nabla}_{\text{Riem. connection}}$

Geodesics & Exponential Map

Idea: geodesics = "straight lines" in a curved space (M, g).

"length-minimizing"? "zero acceleration"?

Defⁿ: A curve $c : I \to M$ is a geodesic in (M, g) if $c'(t)$ is parallel along c, i.e. $\boxed{\frac{Dc'}{dt} = \nabla_{c'} c' \equiv 0}$ — $(*)$



Locally, $(*)$ can be expressed as geodesic eqⁿ

$$\underbrace{\frac{d^2 c_k}{dt^2}(t) + \sum_{i,j} \Gamma_{ij}^k (c(t))\, c_i'(t)\, c_j'(t) = 0}_{\text{2nd order NON-Linear ODE system}}, \quad \forall k = 1, \dots, n$$

ODE theory $\Longrightarrow$ short-time existence & uniqueness with initial data: $c(0)$, $c'(0)$

Note: Since c' is parallel, $g(c'(t), c'(t)) \equiv$ const.

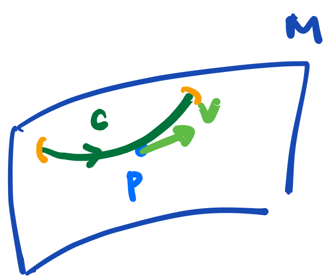
c is p.b.a.l. means $g(c'(t), c'(t)) \equiv 1$

Thm: Given $p \in M$, and $v \in T_pM$, $\exists$ (uniqueness) smooth curve

$$c_{p,v}(t) : (-\varepsilon, \varepsilon) \longrightarrow M$$

s.t. $c_{p,v}$ is a geodesic on M with

$$c_{p,v}(0) = p \qquad c'_{p,v}(0) = v$$



Moreover, the curve $c_{p,v}$ depends smoothly on the initial data p and v. and the interval of existence ε

(homogenity)

Prop: $c_{p,\lambda v}(t) = c_{p,v}(\lambda t)$ for any $\lambda > 0$

(whenever the solutions are defined)

Proof: $\frac{d}{dt}(c_{p,v}(\lambda t)) = \lambda\, c'_{p,v}(\lambda t)$

$\frac{d^2}{dt^2}(c_{p,v}(\lambda t)) = \lambda^2\, c''_{p,v}(\lambda t)$

Check: $c_{p,v}(0) = p$ & $\left.\frac{d}{dt}\right|_{t=0}(c_{p,v}(\lambda t)) = \lambda v$

and $(*)$ is satisfied.

$\square$

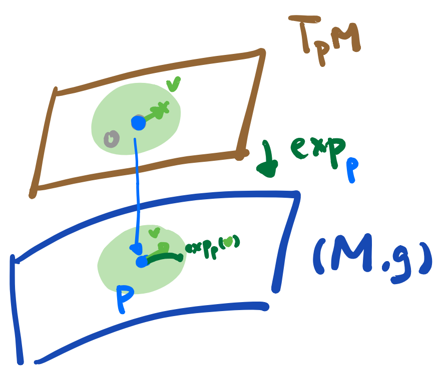
This implies that for any $p \in M$, $\exists$ nbd $\mathcal{U}$ of $0 \in T_pM$ s.t.

$$c_{p,v}(t) \text{ is defined } \forall t \in (-2, 2)$$

Def^{n}: The exponential map of (M, g) at p is

$$\exp_p : \overset{o \subseteq}{\mathcal{U}} \subseteq T_pM \longrightarrow M$$

$$\exp_p(v) := c_{p,v}(1)$$



Prop: $\exp_p$ is a local diffeo. at $0 \in T_pM$

Proof: Smooth dependence of $c_{p,v}$ on v $\Rightarrow$ $\exp_p$ smooth.

Clearly, $\exp_p(0) := c_{p,0}(1) = p$

Claim: $(d\exp_p)_0 = id_{T_pM} : T_pM \to T_pM$ (Note: $T_0(T_pM) \cong T_pM$)

$$(d\exp_p)_0(v) := \left.\frac{d}{dt}\right|_{t=0} \exp_p(tv) = \left.\frac{d}{dt}\right|_{t=0} c_{p,tv}(1)$$

$$\overset{\text{homogenity}}{=} \left.\frac{d}{dt}\right|_{t=0} c_{p,v}(t) = v$$

By I.V.F. Prop follows. $\square$

More generally, we can consider the exponential map

$$\begin{aligned} \exp : \tilde{\mathcal{U}} \subseteq TM &\longrightarrow M \\ (p, v) &\longmapsto \exp_p(v) \end{aligned}$$

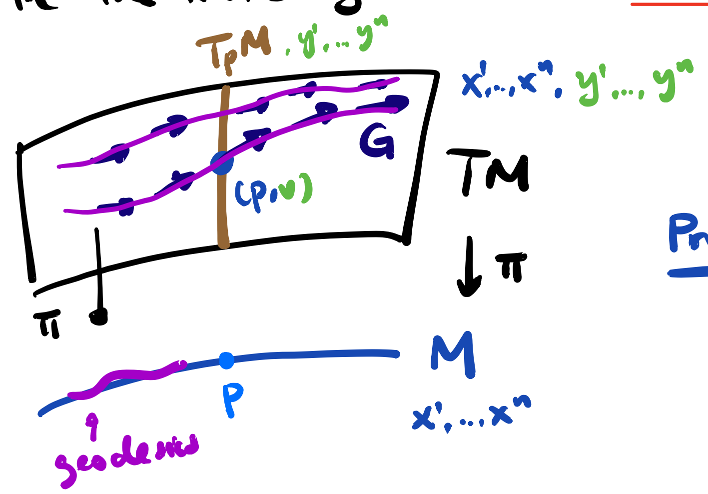
We can view the geodesic eq$^{\underline{n}}$ (*) as a 1st order ODE system at the level of tangent bundle:

locally: $TM \ni (p, v) \approx (\underbrace{x^1, \dots, x^n}_{p}, \underbrace{y^1, \dots, y^n}_{v = \sum_j y^j \frac{\partial}{\partial x^j}})$

$$(*) \iff \begin{cases} \dfrac{dx^k}{dt} = y^k \\ \dfrac{dy^k}{dt} = -\sum_{i,j} \Gamma_{ij}^k(x)\, y^i y^j \end{cases}$$

(R.H.S. is indep of t!)

i.e the R.H.S. gives us a time-indep vector field G on TM.



generates

geodesic flow $\{\varphi_t\} \subset \mathrm{Diff}(TM)$

Prop: The integral curves for this flow projects down to geodesics on M.

Ex: Prove this.

Example 1: $(M^n, g) = (\mathbb{R}^n, g_{\text{Eucl.}})$

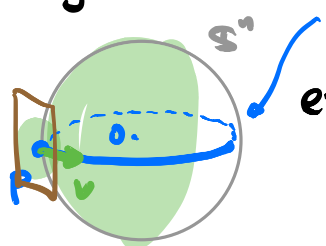
geodesics = straight lines (w/. const speed)

$$\exp_p(v) = p + v$$

$$c_{p,v}(t) = p + tv$$

Example 2: $(M^n, g) \cong (S^n, g_{\text{round}})$

geodesics = "great circles"



$\exp_p : B_\pi(0) \subseteq T_pM \to S^n \setminus \{-p\}$ is diffeomorphism.